stamps for $d=2$, curves *3* and *4* for $d=2.5$ and the values of the remaining parameters are $l=4, d_0=12.5, d_1=10$.

REFERENCES

1. FOMIN V.M., A non-stationary dynamic periodic contact problem for a homogeneous elastic half-plane, PMM, 48, 2, 1984.
2. KAPLAN I.G., Symmetry of Multi-electron Systems. Nauka, Moscow, 1969.
3. PETRASHEN M.I. and TRIFONOV E.D., Application of Group Theory in Quantum Mechanics. Nauka, Moscow, 1967.
4. BURYSHKIN M.L., Expansion of vector-functions defined in a domain with a spatial symmetry group in the translational truncated case. Dokl. Akad. Nauk UzbSSR, Ser.A, 7, 1975.
5. BURYSHKIN M.L., A general scheme for solving inhomogeneous linear problems for symmetrical mechanical systems. PMM, 45, 5, 1981.
6. VLADIMIROV V.S., Generalized Functions in Mathematical Physics. Nauka, Moscow, 1979.
7. SEIMOV V.M., Dynamic Contact Problems. Naukova Dumka, Kiev, 1976.
8. KAGAN V.F., Fundamentals of the Theory of Determinants. Gosizdat Ukrainy, Odessa, 1922.

Translated by M.D.F.

PMM U.S.S.R.,Vol.50,No.3,pp.357-361,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON THE NON-LINEAR BOUNDARY EQUATIONS OF THE MECHANICS OF THE CONTACT OF ROUGH ELASTIC BODIES*

B.A. GALANOV

As a generalization of the results of /1/, the existence and uniqueness of the solution of the contact problem of several rough elastic bodies (in the example of the contact of three bodies is proved). The method of non-linear boundary equations is used, which (like the variational method /2-5/) enables an effective investigation to be made of the correctness of the problem of body contact with unknown contact domains.

1. Formulation of the problem. Let $Oxyz$ be a Cartesian rectangular coordinate system, $M(x, y)$ a point in the plane $E_2=\{z=0\}$ with the coordinates x, y, $\text{mes}\{\omega\}$ the Lebesgue measure of the set $\omega\subset E_2$; Δ^2 the biharmonic operator, $\text{supp}\, v(M)$ a carrier of the function $v(M)$, Q an operator setting the function $v(M)$, $M\in\Omega$ in correspondence with the function $v^+(M)$, $M\in\Omega$ according to the rule $v^+\equiv Qv=\sup\{v(M), 0\}$, and $L_{r,2}=L_{r,2}(\Omega)$ a Banch space of the vector-functions $v(M)=(v_1(M), v_2(M))$ (defined in the domain $\Omega\subset E_2$) with the norm

$$\|v(M)\|=\Big(\int_\Omega(|v_1(M)|^r+|v_2(M)|^r dS_M\Big)^{1/r},\quad r\geqslant 1$$

For $r=2$ the space $L_{r,2}$ is a Hilbert space with the scalar product

$$(u, v)=\int_\Omega(u_1(M)v_1(M)+u_2(M)v_2(M))\,dS_M$$

The linear operator K acting from the Banach space E on the conjugate space E^* to E is called strictly positive if $(Kv, v)\geqslant 0$, and the equality $(Kv, v)=0$ is possible if and only if $v=0$ /6/ ((u, v) is the value of the linear continuous functional $u\in E^*$ at the element $v\in E$).

We will consider the (frictionless) contact problem of an elastic body *1* and elastic half-space *3* with a plate *2* located between them (see the sketch, the y axis is perpendicular to the plane of the sketch). As a boundary value problem it reduces (with known assumptions) to the construction of the respective harmonic functions $u_1(x, y, z)$, $u_3(x, y, z)$ in the half-spaces

Prikl.Matem.Mekhan.,50,3,470-474,1986

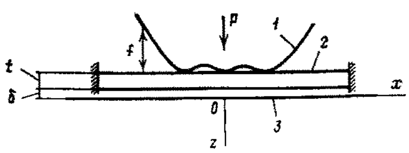
$z < -t-\delta$ and $z > 0$, $(u_i(x,y,z) = O(R^{-1})$ for $R \to \infty$, $R = (x^2+y^2+z^2)^{1/2}$, $i = 1, 3)$ of the solution $u_2(x,y)$ of the equation

$$\Delta^2 u_2(x,y) = \lambda_2 (u_{1z}'(x,y,-t-\delta) - u_{3z}'(x,y,0)), M(x,y) \in \Omega$$

and to the determination of the plane closed domains $S_1 \subset \Omega$, $S_2 \subset \Omega$ and the scalar quantity h from the conditions

$$\begin{gathered} z=-t-\delta,\ \Phi_1(u_{1z}') + 2\pi\lambda_1 u_1 + u_2 = q(M),\ u_{1z}' \geqslant 0 \quad (M(x,y) \in S_1) \\ \Phi_1(u_{1z}') + 2\pi\lambda_1 u_1 + u_2 > q(M),\ u_{1z}' = 0\ (M(x,y) \in (\Omega \setminus S_1)) \\ u_{1z}' = 0\ (M(x,y) \in (E_2 \setminus \Omega)) \quad (q(x,y) = h - f(x,y) \in L_r(\Omega)) \end{gathered} \tag{1.1}$$

$$\begin{gathered} z=0,\ \Phi_2(u_{3z}') + 2\pi\lambda_3 u_3 - u_2 = -\delta,\ u_{3z}' \geqslant 0\ (M(x,y) \in S_2) \\ \Phi_2(u_{3z}') + 2\pi\lambda_3 u_3 - u_2 > -\delta,\ u_{3z}' = 0\ (M(x,y) \in (\Omega \setminus S_2)) \\ u_{3z}' = 0\ (M(x,y) \in (E_2 \setminus \Omega)) \\ \iint_{S_1} u_{1z}'(x,y,-t-\delta)\,dx\,dy = P, \quad u_2|_\Gamma = \frac{\partial u_2}{\partial n}\Big|_\Gamma = 0 \end{gathered}$$



Here $\Omega \subset E_2$ is a bounded domain with boundary Γ, n is the unit vector of the external normal to the contour Γ, u_{iz}' is the derivative of the function u_i $(i = 1, 3)$ with respect to z, $\Phi_i(p_i)$, $-\infty < p_i < \infty$ $(i = 1, 2)$ are strictly increasing continuous functions and $\Phi_i(0) = 0$, $\lambda_i > 0$ $(i = 1, 2, 3)$ are constants, $t > 0$ is the plate thickness, $\delta \geqslant 0$ is the gap between the plate *2* and the half-space *3*, and $h > 0$ is a rigid displacement of the body *1* along the z axis caused by a force $P > 0$ acting on the body *1*. The function $f(x,y) \geqslant 0$ determines the gap between the plate the body *1* until their deformation (sketch), i.e., for $h = 0$ and $P = 0$. The orthogonal projections of the sets S_1 and S_2, on the planes $z = -t-\delta$ and $z = -\delta$, respectively, are those parts of the plate surface that come into contact with the body *1* and half-space *3* after deformation. The functions $v_i = \Phi_i(p_i)$ $(i = 1, 2)$ describe the mechanical characteristics of the roughness of the bodies. Henceforth, $p_i = F_i(v_i)$ denotes the inverse function for $v_i = \Phi_i(p_i)$.

If the potentials

$$u_1(x,y,z) = \frac{1}{2\pi}\iint_{S_1} \frac{p_1(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+(z+t+\delta)^2]^{1/2}},$$
$$z \leqslant -t-\delta$$
$$u_2(x,y) = \lambda_2 \iint_{S_1} G(x,y,\xi,\eta)\,p_1(\xi,\eta)\,d\xi\,d\eta - \lambda_2 \iint_{S_2} G(x,y,\xi,\eta)\,p_2(\xi,\eta)\,d\xi\,d\eta$$
$$u_3(x,y,z) = \frac{1}{2\pi}\iint_{S_2} \frac{p_2(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{1/2}}, \quad z \geqslant 0$$

are introduced ($G(x,y,\xi,\eta)$ is Green's function for the boundary value problem $\Delta^2 w(M) = 0$, $M \in \Omega$; $w|_\Gamma = \partial w/\partial n|_\Gamma = 0$) and we use the notation

$$H(M,N) = [(x-\xi)^2 + (y-\eta)^2]^{-1/2}$$
$$Hp_i = \int_{S_i} H(M,N)\,p_i(N)\,dS_N, \quad Gp_i = \int_{S_i} G(M,N)\,p_i(N)\,dS_N$$
$$Ip_1 = \int_{S_1} p_1(N)\,dS_N; \quad M(x,y) \in \Omega, \quad N(\xi,\eta) \in S_i, \quad i = 1, 2$$

then problem (1.1) is eqüivalent to the problem of seeking the contact pressures $p_1(M)$, $p_2(M)$, the sets S_1, S_2 and the quantity h from the system

$$\begin{aligned}
&\Phi_1(p_1) + (\lambda_1 H + \lambda_2 G) p_1 - \lambda_2 G p_2 = q; \quad p_1 \geqslant 0 \quad (M \in S_1) \\
&\Phi_1(p_1) + (\lambda_1 H + \lambda_2 G) p_1 - \lambda_2 G p_2 > q; \; p_1 = 0 \; (M \in (\Omega \setminus S_1)) \\
&\Phi_2(p_2) - \lambda_2 G p_1 + (\lambda_3 H + \lambda_2 G) p_2 = -\delta; \quad p_2 \geqslant 0 \quad (M \in S_2) \\
&\Phi_2(p_2) - \lambda_2 G p_1 + (\lambda_3 H + \lambda_2 G) p_2 > -\delta; \; p_2 = 0 \; (M \in (\Omega \setminus S_2)) \\
&I p_1 = P
\end{aligned} \tag{1.2}$$

We introduce the vectors and the symmetric second-order matrix

$$\begin{aligned}
&g(M) = (q(M), -\delta), \; v(M) = (v_1(M), v_2(M)), \; F(v(M)) = \\
&\quad (F_1(v_1(M)), F_2(v_2(M))), \; Q(F(v(M))) = (Q(F_1(v_1(M))), \\
&Q(F_2(v_2(M)))) \\
&K(M,N) = \begin{Vmatrix} \bar{\lambda}_1 H(M,N) + \varkappa & -\varkappa \\ -\varkappa & \bar{\lambda}_3 H(M,N) + \varkappa \end{Vmatrix} \\
&\varkappa = \bar{\lambda}_2 G(M,N), \; \bar{\lambda}_i = \lambda_i / \lambda, \; \lambda = \lambda_1 + \lambda_2 + \lambda_3, \; i = 1, 2, 3
\end{aligned}$$

We consider the system of non-linear equations

$$\begin{aligned}
&v(M) + \lambda \int_\Omega K(M,N) Q(F(v(N))) \, dS_N = g(M) \\
&\int_\Omega Q(F_1(v_1(N))) \, dS_N = P; \quad M, N \in \Omega
\end{aligned}$$

in the unknown vector function $v(M)$ and the scalar h, which we will write in operator form for convenience

$$v + \lambda KQFv = g, \; IQF_1 v_1 = P \tag{1.3}$$

Theorem 1. If (v_1^*, v_2^*, h) is a solution of system (1.3), then $(p_1 = QF_1 v_1^*, \; p_2 = QF_2 v_2^*,$ $S_1 = \{M: v_1^* \geqslant 0\}, \; S_2 = \{M: v_2^* \geqslant 0\}, \; h)$ is a solution of system (1.2), where $\operatorname{supp} p_i(M) \subseteq S_i$, $i = 1, 2$. Conversely, if (p_1, p_2, S_1, S_2, h) is a solution of system (1.2) and

$$v^* = g - \lambda KQp \quad (M \in \Omega, N \in S_i) \tag{1.4}$$

then the triple (v_1^*, v_2^*, h) is a solution of system (1.3). The domains S_1 and S_2 can be multi-connected.

Proof. To reduce the writing, we use the notation

$$\begin{aligned}
&\omega_1 = S_1 \cap S_2; \; \omega_2 = S_2 \cap (\Omega \setminus S_1); \; \omega_3 = S_1 \cap (\Omega \setminus S_2); \\
&\omega_4 = \Omega \setminus (S_1 \cup S_2)
\end{aligned}$$

Let (v_1^*, v_2^*, h) be the solution of system (1.3) and $M \in \omega_1$. The relationships

$$\begin{aligned}
&p_1 = QF_1 v_1^* \geqslant 0, \; p_2 = QF_2 v_2^* \geqslant 0, \; \Phi_1(p_1) + (\lambda_1 H + \\
&\lambda_2 G) p_1 - \lambda_2 G p_2 = q, \; \Phi_2(p_2) - \lambda_2 G p_1 + (\lambda_3 H + \lambda_2 G) p_2 = \\
&-\delta, \; I p_1 = P
\end{aligned}$$

meaning that (p_1, p_2, S_1, S_2, h) is a solution of system (1.2) for $M \in \omega_1$, result from the definition of Q and (1.3). It can similarly be shown that (p_1, p_2, S_1, S_2, h) is a solution of system (1.2) for $M \in \omega_i$, $i = 2, 3, 4$.

The inclusions $\operatorname{supp} p_i(M) \subseteq S_i$ $(i = 1, 2)$ are obvious. This completes the proof of the direct part of the theorem.

If (p_1, p_2, S_1, S_2, h) is a solution of system (1.2), then it can be seen (by alternately considering the case $M \in \omega_i$, $i = 1, 2, 3, 4$) that (1.4) can be written thus: $v^* = g - \lambda KQFv^*$, i.e. $v^* = (v_1^*, v_2^*, h)$ is a solution of (1.3).

Therefore, to solve the above contact problem (1.2) it is sufficient to find the solution (v_1^*, v_2^*, h) of system (1.3) since $p_1 = QF_1 v_1^*$, $p_2 = QF_2 v_2^*$, $S_1 = \{M: v_1^* > 0\}$, $S_2 = \{M: v_2^* > 0\}$. Consequently we will henceforth investigate system (1.3).

2. Correctness of the problem. Let $v^\circ(M) = (v_1^\circ, (M), v_2^\circ(M))$ be a solution of the system of Hammerstein Eqs.(1.3) for $h = h_0$, and let

$$P_0 = \int_\Omega Q(F_1(v_1^\circ(M))) \, dS_M \tag{2.1}$$

We further assume that the functions F_i $(i = 1, 2)$ satisfy the condition

$$|F_i(v)| \leqslant c_* |v|^{1/\alpha}; \; c_* = \text{const}, \; 0 < \alpha \leqslant 1$$

Theorem 2. If $r = 1 + 1/\alpha$, $1/2 < \alpha \leqslant 1$ and $P \in [0, P_0]$, then system (1.3) has the unique solution $\{v^* \in L_{r,2}; h \in [0, h_0]\}$. Here $h = h(P)$ $(0 \leqslant P \leqslant P_0)$ is a continuous strictly increasing function, and if the function $f(M)$ is continuous, then $v^*(M) \in C(\Omega)$ (h_0 is an arbitrary number from the range $[0, \infty)$. The quantity P_0 defined by (2.1) corresponds to this number. The quantity $C(\Omega)$ is the space of vector functions whose coordinates are continuous functions on Ω).

Proof. The operator K is a completely continuous operator from $L_{q,2}$ $(q = 1 + \alpha)$ in $L_{q,2}{}^* = L_{r,2}$ $(r = 1 + 1/\alpha)$ /7/. The contraction K in $L_{2,2}$ is a selfadjoint strictly positive operator. Consequently, there exists a square root $D = K^{1/2}$ of the operator K that is a completely continuous operator from $L_{2,2}$ into $L_{q,2}{}^*$ /6/. The adjoint operator D^* acts from $L_{q,2}{}^*$ into $L_{2,2}$.

If the change of variable $v = Dt + g$ /6/ is made in the first equation of system (1.3), we obtain instead the equivalent equation

$$Ut \equiv t + \lambda D^* QF(Dt + g) = 0;\ t \in L_{2,2} \tag{2.2}$$

with a continuous monotonic and potential operator U (the monotonicity of U follows from the monotonicity of the function $QF_i(v_i)$ $(i = 1, 2)$).

We find the lower bound of the scalar product (Ut, t):

$$(Ut, t) = (t, t) + \lambda (QF(Dt + g), Dt + g) - \lambda (QF(Dt + g), g) \geqslant (t, t) - \lambda (QF(Dt + g), g) \geqslant (t, t) - \lambda \| g \|_{L_{r,2}} \| QF(Dt + g) \|_{L_{q,2}}$$

Using the properties of the operators Q and F_i as well as the Minkowski inequality, we obtain

$$\| QF(Dt + g) \|_{L_{q,2}} \leqslant c_* (\| D \| \| t \|_{L_{2,2}} + \| g \|_{L_{r,2}})^{1/\alpha}$$

We hence have the estimate

$$(Ut, t) \geqslant \| t \|^2_{L_{2,2}} - c_* \lambda \| g \|_{L_{r,2}} \| t \|^{1/\alpha}_{L_{2,2}} (\| D \| + \| g \|_{L_{r,2}} / \| t \|_{L_{2,2}})^{1/\alpha}$$

and a number $\rho > 0$ exists for $\alpha > 1/2$ such that the inequality $(Ut, t) > 0$ holds for $\| t \|_{L_{2,2}} \geqslant \rho$, i.e., on the basis of the Browder-Minty theorem (/6/, p.262), Eq.(2.2) has the solution $t^* \in L_{2,2}$ for all $h \in [0, \infty)$, to which the solution $v^* = Dt^* + g$ of the first equation of (1.3) corresponds.

Hence, the existence of the function

$$P(h) = \int_\Omega Q(F_1(v_1^*(M)))\, dS_M; \quad h \in [0, \infty)$$

follows directly.

We show the strict monotonicity and continuity of this function.

Let the vector-functions v_1, v_2 be solutions of the first equation of (1.3) corresponding to the values $h = h_1$ and $h = h_2$. We introduce the notation

$$\varepsilon = v_2 - v_1,\ d = QFv_2 - QFv_1,\ \Delta h = (h_2 - h_1, 0)$$

We then obtain from (1.3)

$$\begin{gathered} \varepsilon + \lambda K d = \Delta h \\ (\varepsilon, d) + \lambda (Kd, d) = (\Delta h, d) \equiv (h_2 - h_1)(P(h_2) - P(h_1)) \end{gathered} \tag{2.3}$$

where $(\varepsilon, d) \geqslant 0$ (a consequence of the monotonicity of the operator QF).

Using the assumption that the set $\{M: f(M) = 0\} \neq \varnothing$, it can be proved (by reductio absurdum) that $d \neq 0$ for $h_1 \neq h_2$. Hence, it follows from the strict positivity of K (in the sense of the definition given in the introduction) and from (2.3) that $d = 0$ just for $h_1 = h_2$. Consequently, for $h_1 \neq h_2$ the left side of the second equation in (2.3) is positive. This implies the strict increase of the function $P(h)$. Furthermore, from the strict positivity of K and the second equation in (2.3) it follows that $d \to 0$ as $\Delta h \to 0$ and $P(h)$ is a continuous function. Consequently, also the function $h = h(P)$ $(P \in [0, P_0])$ is a strictly increasing continuous function with the domain of values $[0, h_0]$.

The uniqueness of the solution of system (1.3) is an obvious consequence of equalities (2.3) and the fact that $d = 0$ only for $\Delta h = 0$.

We will now show that $v^* \in C(\Omega)$ for $g \in C(\Omega)$. We have

$$v^* + \lambda KQFv^* = g \tag{2.4}$$

An alternative is possible: 1) $v^* \in L_{r,2}$ is a discontinuous bounded vector-function; 2) $v^* \in L_{r,2}$ is an unbounded vector-function. For the first possibility, the left side of inequality (2.4) is a discontinuous bounded vector-function (since the operator λKQF converts every bounded vector function into a continuous vector-function), which contradicts the continuity of g. If it is taken into account that the function H has a weak singularity while the function G is continuous, then the second possibility results in an analogous contradiction. Consequently v^* is a continuous vector-function.

Other conditions for the existence of a solution of system (1.3) can be mentioned. For instance, for $g \in C(\Omega)$ $(0 < \alpha \leqslant 1)$ and sufficiently small λ_i the Schauder principle can be used, as is done in /1/.

Different approximate methods /6, 8/ can be used to solve system (1.3). It should be taken into account here that the operator Q is Fréchet-differentiable only in certain sets $\omega \subset L_p$. For instance, (as an operator acting from $C(\Omega) \subset L_p(\Omega)$ into $L_p(\Omega)$, it is differentiable in the set

$$\omega = \{v : v \in C(\Omega),\ \mathrm{mes}\,\{M : v(M) = 0\} = 0\}$$

REFERENCES

1. GALANOV B.A., Spatial contact problems for elastic rough bodies during elastic-plastic deformations of the roughness. PMM, 48, 6, 1984.
2. GLOWINSKI R., LIONS J.-L and TREMOLIERE R., Numerical Investigation of Variational Inequalities. Mir, Moscow, 1979.
3. FICHERA G., Existence Theorems in Elasticity Theory. Mir, Moscow, 1974.
4. KRAVCHUK A.S., Formulation of the contact problem for several deformable bodies as a non-linear programming problem, PMM, 42, 3, 1976.
5. RABINOVICH V.L. and SPEKTOR A.A., Solution of certain classes of spatial contact problems with unkown boundaries. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 2, 1985.
6. VAINBERG M.M., The variational Method and the Method of Monotonic Operators in the Theory of Non-linear Equations. Nauka, Moscow, 1972.
7. MIKHLIN S.G., Linear Partial Differential Equations. Vyssh. Shkola, Moscow, 1977.
8. KRASNOSEL'SKII M.A., VAINIKKO G.M., ZABREIKO P.P., RUTITSKII YA.V. and STETSENKO V.YA., Approximate Solution of Operator Equations. Nauka, Moscow, 1969.

Translated by M.D.F.

PMM U.S.S.R., Vol.50, No.3, pp.361-367, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

INVERSE CONTACT PROBLEMS OF THE THEORY OF PLASTICITY*

V.I. KUZ'MENKO

A class of inverse contact problems of the theory of plasticity dealing with the determination of the form of a stamp ensuring the prescribed final change in the body shape is studied. The problem is given in the form of a functional equation. The principle of compressive mapping is used to show the existence and uniqueness of the solution, and an iterative process is given for determining the required form of the stamp. A problem dealing with the form of the stamp ensuring the formation of trapezoidal indentations in the strip surface is solved as an example.

1. Formulation of the problem. We shall connect a monotonically increasing parameter t, $t \in [0, T]$, with the process of quasistatic deformation of an elastoplastic body Ω, and we shall call it time. We use, as the spatial frame of reference, the Cartesian coordinate system $0x_1x_2x_3$. The symbols $u_i(x, t)$, $\varepsilon_{ij}(x, t)$, $\sigma_{ij}(x, t)$ denote the components of the vector of small displacements and of the small deformations and stress tensors at the point $x = (x_1, x_2, x_3)$, at the instant t.

The body Ω is bounded by a piecewise smooth surface composed of three parts: Γ_u, Γ_σ, Γ_c. The body is clamped over the surface Γ_u and the part Γ_σ is stress-free. The surface Γ_c is acted upon by the moving stamp. We describe the form of the stamp surface by the function $f(x)$ equal to the distance from the surface Γ_c to the stamp surface along the normal to Γ_c, at $t = 0$. The law of motion of the stamp as a rigid body is assumed given, does not depend on the form of the stamp, and must be chosen so that when $t < t^*$, an elastoplastic deformation takes place in the body Ω, while at $t > t^*$ we only have unloading or active elastic deformation. We assume that there is no contact whatsoever between the body and the stamp at $t = T$.

*Prikl.Matem.Mekhan., 50, 3, 475-482, 1986